1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

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2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
1. Let \( X_1, X_2, X_3, X_4 \) be a random sample of size 4 from the normal distribution with mean 0 and variance 1.

(a) Find the moment generating function of \( W = X_1 \times X_2 \).
(b) Find the moment generating function of \( V = X_1 \times X_2 - X_3 \times X_4 \).
(c) Using the moment generating function or other method, prove that \( V \) in (b) has a double exponential distribution with p.d.f. \( g(v) = \frac{1}{2}e^{\left| v \right|}, \quad -\infty < v < \infty \).

Hint: You can use the following facts and other known distributional properties (such as the relation among normal, \( \chi^2 \), exponential, and gamma distributions) without proof: (1) Let \( X \sim N(\mu, \sigma^2) \) with \( E(X) = \mu \) and \( Var(X) = \sigma^2 \). Then the moment generating function of \( X \) is \( e^{\mu t + \frac{1}{2}\sigma^2 t^2} \). (2) Let \( Y \sim Gamma(\alpha, \beta) \) with \( E(Y) = \alpha\beta \) and \( Var(Y) = \alpha\beta^2 \). Then the moment generating function of \( Y \) is \( \left( \frac{1}{1-\beta t} \right)^\alpha \).
2. Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a Bernoulli distribution with probability of success \( \theta \), where \( 0 < \theta < 1 \).

(a) Show that \( T = \sum_{i=1}^{n} X_i \) is a complete sufficient statistics for \( \theta \).

(b) Find UMVUE of \( \theta^r (1 - \theta)^s \), where \( r \) and \( s \) are non-negative integers with \( 1 \leq r + s < n \).

(c) Is there unbiased estimator of the odd ratio, \( \frac{\theta}{1-\theta} \)? Justify your answer.
3. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size $n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where $x_1, x_2, \cdots, x_n$ are fixed known constants and $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ are iid with $N(0, \sigma^2)$, $\sigma^2$ unknown.

We consider three estimators of $\beta$: MLE of $\beta$, $\hat{\beta}_{MLE}$, $\hat{\beta}_a = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}$, and $\hat{\beta}_b = \frac{\sum_{i=1}^n Y_i/x_i}{n}$.

(a) Find the MLE of $\beta$, $\hat{\beta}_{MLE}$.

(b) Prove the all three estimators of $\beta$ are unbiased.

(c) Compare these three estimators of $\beta$ in terms of their variances.
4. Let \((X_1, Y_1) \ldots, (X_n, Y_n)\) be a random sample of size \(n\) from a bivariate normal distribution with unknown parameters: means \(\mu_1, \mu_2\), variances \(\sigma_1^2, \sigma_2^2\) and correlation coefficient \(\rho\).

(a) Describe the paired \(t\) test procedure for testing \(H_0 : \mu_1 = \mu_2\) vs. \(H_1 : \mu_1 \neq \mu_2\).

(b) Derive the likelihood ratio test for testing \(H_0 : \mu_1 = \mu_2\) vs. \(H_1 : \mu_1 \neq \mu_2\).

(c) Argue that the paired \(t\) test in (a) and the LRT in (b) are equivalent.
5. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from independent Exponential distributions with means $\mu$ and $\lambda \mu$ respectively.

(a) Find the MLE of $P(X_1 > Y_1)$ and hence find the MLE of $P(X_{(1)} > Y_{(1)})$, where $X_{(1)} = \min(X_1, \ldots, X_m)$ and $Y_{(1)} = \min(Y_1, \ldots, Y_n)$.

(b) Find jointly sufficient statistics $S$ for $(\lambda, \mu)$.

(c) Prove or disprove that $S$ jointly minimally sufficient for $(\lambda, \mu)$. 
6. Consider the linear model
\[ Y = X\beta + \epsilon, \]
where \( Y \) is \((n \times 1)\), \( \epsilon \) is \((n \times 1)\), \( X \) is \((n \times p)\), and where \( \epsilon \sim N(0, \sigma^2 I) \).

(a) Let \( A \) be a symmetric \( n \times n \) matrix. Prove that under the null hypothesis \( H_0 : \beta = 0 \),
\[ \frac{Y'AY}{\sigma^2} \sim \chi^2_p. \]
if and only if \( A \) is an idempotent matrix such that \( \text{trace}(A) = p \).

(b) Give a comprehensive explanation of how you the result of part a) can be used to develop tests of hypotheses in an analysis of variance.
7. Suppose that $X|n, \theta$ has a binomial distribution with parameter $\theta$. Suppose we put independent prior distributions on $n$ and $\theta$, with $n$ having Poisson($\lambda$) prior and $\theta$ having a Beta($\alpha, \beta$) prior, where $\alpha$ and $\beta$ are known hyperparameters.

(a) Prove that the posterior density of $\theta$ given $X = x$ and $n$ is Beta($x + \alpha, n - x + \beta$).
(b) Prove that the posterior probability function of $n + X$ given $X = x$ and $\theta$ is Poisson\[(1 - \theta)\lambda].\]
(c) Suppose $\alpha = \beta = 1$ and $X = 10$, explain in details how you can obtained 100 samples of $n$’s from the posterior distribution of $n$ given $X = 10$. 


8. Let $X$ and $Y$ be random variables such that $Y|X = x \sim \text{Poisson}(\lambda x)$, and $X$ has density
$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

(a) Prove that
i. $E(Y) = \lambda$ and $\text{Var}(Y) = \lambda + \theta \lambda^2$.
ii. $Y$ has density
$$f_Y(y; \lambda) = \frac{\Gamma(\theta + y) \lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

(b) Now suppose that $Y_1, \ldots, Y_n$ are independent random variables from the distribution given above, with $Y_i$ having mean $\lambda_i$, and $\log(\lambda_i) = \beta z_i$, where $z_i$'s are known covariates, $i = 1, \ldots, n$, and assume that $\theta = 1$. Write a Fisher scoring algorithm for computing the MLE of $\beta$, and discuss its properties.
9. Let \( \{X_1, \ldots, X_n\} \) be a random sample from the normal distribution with mean \( \mu_1 \) and variance \( \sigma_x^2 \) and \( \{Y_1, \ldots, Y_m\} \) a random sample from the normal distribution with mean \( \mu_2 \) and variance \( \sigma_y^2 \). Assume that the \( X_i \)’s are independently distributed of the \( Y_j \)’s. Put
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad S_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i, \quad S_y^2 = \frac{1}{m-1} \sum_{i=1}^{m} (Y_i - \bar{Y})^2,
\]
\( \hat{\sigma}^2 = \frac{1}{n} S_x^2 + \frac{1}{m} S_y^2. \)

(a) What are \( a \) and \( b \) if one approximates the sampling distribution of \( \hat{\sigma}^2 \) by \( a\chi^2_b \), where \( \chi^2_b \) is a central Chi-square random variable with degrees of freedom \( b \).

(b) Derive a \((1 - \alpha)\) % approximate confidence interval for \( \mu_1 - \mu_2 \) by using the approximation in (a).
10. Let \{X_1, \ldots, X_n\} be a random sample from the population with density \( f(x; \theta_i, i = 1, 2) \), where \( f(x; \theta_i, i = 1, 2) \) is given by:

\[
f(x; \theta_i, i = 1, 2) = \begin{cases} 
\frac{1}{\theta_2 - \theta_1} & \text{if } \theta_1 \leq x \leq \theta_2, \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Show that the statistics \( \{X_{(1)} = \min(X_1, \ldots, X_n), X_{(n)} = \max(X_1, \ldots, X_n)\} \) are sufficient and complete statistics for the parameters \( \{\theta_i, i = 1, 2\} \).

(b) Derive the UMVUE of \( \theta_2 - \theta_1 \).
11. Let \( \{X_1, \ldots, X_n\} \) be a random sample from the population with density \( f(x; \theta, \mu_i, i = 1, 2) = \theta f_1(x; \mu_1) + (1 - \theta) f_2(x; \mu_2) \), where \( f_i(x; \mu_i) \) is the density of the normal distribution with mean \( \mu_i \) and variance 1, and \( 0 < \theta < 1 \). Illustrate how to derive a procedure to compute the MLE (Maximum Likelihood Estimator) of \( \{\theta, \mu_i, i = 1, 2\} \) by using the EM-algorithm.
12. Let \( \{X_1, \ldots, X_m\} \) be a random sample from the normal population with mean \( \mu_1 \) and variance \( \sigma_1^2 \). Let \( \{Y_1, \ldots, Y_n\} \) be a random sample from the normal population with mean \( \mu_2 \) and variance \( \sigma_2^2 \) independently of \( \{X_1, \ldots, X_m\} \).

(a) Derive the size \( \alpha \) Likelihood Ratio test for testing \( H_0 : \sigma_1^2 = \sigma_2^2 \) vs \( H_1 : \sigma_1^2 \neq \sigma_2^2 \).

(b) Derive the power function of your test.

(c) Derive a \( 1 - \alpha \) % confidence interval for \( \theta = \sigma_1^2 / \sigma_2^2 \). If you use this confidence interval to test the above hypothesis \( H_0 \), how is this compared with the procedure of (a)?