A Generalization of the Assouad Embedding

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Abstract

Assouad [Ass83] showed that any metric $\mathcal{M}$ of doubling dimension $\dim$ can be embedded into $l^d_{\infty}$ where $d \leq \varepsilon - O(\dim)$ with distortion at most $(1 + \varepsilon)$. If $X \subseteq \mathcal{M}$ is of doubling dimension, we show that we can use the Assouad technique to embed $X$ into $l^d_{\infty}$ such that the distances from $X \cup S$ to $X$ are preserved where $S \subseteq \mathcal{M}$ is an arbitrary but fixed finite set.

1 Modification of Assouad’s embedding

Assouad [Ass83] showed that any metric $\mathcal{M}$ of doubling dimension $\dim$ can be embedded into $l^d_{\infty}$ where $d \leq \varepsilon - O(\dim)$ with distortion at most $(1 + \varepsilon)$. Following the description in [HM05], we obtain the following result:

Theorem 1.1 Given a metric space $(\mathcal{M}, d_{\mathcal{M}})$, a subset $X \subseteq \mathcal{M}$ such that $X$ has doubling dimension $\dim$, and a finite subset $S \subseteq \mathcal{M}$, we can map the metric $(X \cup S, d_{\mathcal{M}})$ into $l^d_{\infty}$ with $d \leq \varepsilon - O(\dim)$ by a mapping $\Phi$ such that for any $x \in X \cup S$ and any $y \in X$ \[ \|\Phi(x) - \Phi(y)\| \leq 1 + O(\varepsilon). \]

The following elementary lemma, which we mention without proof, is used in the proof below.

Lemma 1.2 Let $(\mathcal{M}, d_{\mathcal{M}})$ be any metric space and let $S \subseteq \mathcal{M}$ be a set of points such that for each $x \in \mathcal{M}$, $d(x, S) = \inf\{d(x, y) | y \in S\}$ exists and is attained for some $y \in S$. Then $d(x, S)$ is $1$-lipschitz.

Proof: Given $r > 0$, we first show how to embed $(X \cup S, d_{\mathcal{M}})$ into $l^d_{\infty}$ with $d_1 \leq \varepsilon - O(\dim)$ such that the following hold

1. For any $x \in X \cup S$ and $y \in X$, \[ \|\phi^r(x) - \phi^r(y)\| \leq \min\{r, d_{\mathcal{M}}(x, y)\}. \]
2. For $x \in X \cup S$, $y \in X$ if $d_{\mathcal{M}}(x, y) \in [(1 + \varepsilon)r, 2r]$ then \[ \|\phi^r(x) - \phi^r(y)\| \geq (1 - \varepsilon)r. \]

Let $N^{(r)}$ be an $\varepsilon r$ net of $X$. Suppose we color the points of $N^{(r)}$ such that any two points $p, q$ with $d_{\mathcal{M}}(p, q) \leq 4r$ get colored differently. Since the metric $(\mathcal{M}, d_{\mathcal{M}})$ is of doubling dimension $\dim$, $d_1 = (\frac{4}{\varepsilon})^{\dim} = \varepsilon - O(\dim)$ colors are sufficient for this purpose. For each color $i$ denote the set of points in $N^{(r)}$ with color $i$ by $C_i$ and define the value $\phi^r_i(x) = \max\{0, r - d_{\mathcal{M}}(x, C_i)\}$. It is easy to see that $\phi^r_i(x) \leq r$ and hence
\[ \| \phi^{(r)}(x) - \phi^{(r)}(y) \| \leq r. \] If \( d_M(x, y) > r \) then clearly \( \| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| < d_M(x, y) \) for each \( i \). Let \( d_M(x, y) \leq r \) and \( i \) be a color. If \( \phi^{(r)}_i(x) = 0 = \phi^{(r)}_i(y) \) or \( \phi^{(r)}_i(x) = r - d_M(x, C_i), \phi^{(r)}_i(y) = r - d_M(y, C_i) \), then also \( \| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| \leq d_M(x, y) \). The other two cases are similar and so we consider one of them where \( \phi^{(r)}_i(x) = r - d_M(x, C_i) \) and \( \phi^{(r)}_i(y) = 0 \). Let \( d_M(x, C_i) = d_M(x, p) \) where \( p \in C_i \). Notice that \( r \leq d_M(y, C_i) \leq d_M(y, p) \) Then,

\[
\| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| = r - d_M(x, p) \\
\leq d_M(y, p) - d_M(x, p) \\
\leq d_M(x, y)
\]

In all the cases \( \| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| \leq d_M(x, y) \). Thus \( \| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| \leq d_M(x, y) \).

The other property can be seen as follows. Suppose \( d_M(x, y) \in [(1 + \varepsilon)r, 2r) \) and \( y \in X \). There is a point \( p \) of the net \( N^{(r)} \) such that \( d_M(y, p) \leq \varepsilon r \). Let \( i \) be the color of \( p \). Thus \( \phi^{(r)}_i(y) \geq (1 - \varepsilon)r \). Now \( d_M(x, p) \geq r \) and for any other point \( q \) of color \( i \), \( d_M(q, x) \geq 4r - (2 + \varepsilon)r = (2 - \varepsilon)r \) and so \( d_M(x, C_i) \geq r \). This means \( \phi^{(r)}_i(x) = 0 \) and so in this coordinate \( i \), \( \| \phi^{(r)}_i(x) - \phi^{(r)}_i(y) \| \geq (1 - \varepsilon)r \) and therefore

\[ \| \phi^{(r)}(x) - \phi^{(r)}(y) \| \geq (1 - \varepsilon)r. \]

The rest of the Assouad construction should follow verbatim, but we reproduce it here for completeness. We use the maps \( \phi^{(r)} \) constructed above for various values of \( r \) to embed points in \( X \cup S \) into \( \mathbb{R}^{d_1 d_2} \) with the \( l_\infty \) metric for some number \( d_2 \) depending on \( \varepsilon \). The exact dependence of \( d_2 \) of \( \varepsilon \) will be specified later. For each integer \( k \), let \( \phi_k(x) \) denote the mapping which maps \( x \) to the matrix with \( d_2 \) rows and \( d_1 \) columns where the \( k \mod d_2 \) row is the vector \( \phi^{(1 + \varepsilon)^k}(x) \) as above and the rest of the entries are zero. We define \( \phi(x) \) as

\[
\phi(x) = \sum_{k \in \mathbb{Z}} \frac{\phi_k(x)}{(1 + \varepsilon)^{k/2}}
\]

We now estimate the value of \( \| \phi(x) - \phi(y) \| \). Let \( l_0 \in \mathbb{Z} \) be the unique integer for which \( d_M(x, y) \in [(1 + \varepsilon)^{l_0+1}, (1 + \varepsilon)^{l_0+2}) \). We notice that for the integer \( l_0 \) the map \( \phi_{l_0}(z) \), which is just \( \phi^{(1 + \varepsilon)^{l_0}}(z) \) with more coordinates that are zero, has the scale \( r = (1 + \varepsilon)^{l_0} \) such that \( d_M(x, y) \in [(1 + \varepsilon)r, 2r) \). We choose \( d_2 = 8\varepsilon^{-1} \log(\varepsilon^{-1}) \) in what follows. In the row of the matrix congruent to \( l_0 \mod d_2 \) the difference between \( \phi(x) \) and \( \phi(y) \) can be lower bounded as

\[
\left\| \sum_{k \in \mathbb{Z}} \phi_{l_0+k d_2}(x) - \phi_{l_0+k d_2}(y) \right\|_\infty \geq \left\| \phi_{l_0}(x) - \phi_{l_0}(y) \right\|_\infty - \sum_{k < 0} \left\| \phi_{l_0+k d_2}(x) - \phi_{l_0+k d_2}(y) \right\|_\infty \\
- \sum_{k > 0} \left\| \phi_{l_0+k d_2}(x) - \phi_{l_0+k d_2}(y) \right\|_\infty \\
\geq (1 - \varepsilon)(1 + \varepsilon)^{l_0/2} - \sum_{k > 0} \frac{(1 + \varepsilon)^{l_0+2} - (1 + \varepsilon)^{(l_0+k d_2)/2}}{(1 + \varepsilon)^{(l_0+k d_2)/2}} \\
= (1 - O(\varepsilon))\sqrt{d_M(x, y)}
\]

On the other hand, for any integer \( a \in \{0, 1, \ldots, d_2 - 1\} \) we have for the rows with
index congruent to \((l_0 + a) \mod d_2\)

\[
\left\| \sum_{k \in \mathbb{Z}} \phi_{l_0 + a + kd_2}(x) - \phi_{l_0 + a + kd_2}(y) \right\| \leq \sum_{k \leq 0} \| \phi_{l_0 + a + kd_2}(x) - \phi_{l_0 + a + kd_2}(y) \|_\infty \\
+ \sum_{k > 0} \| \phi_{l_0 + a + kd_2}(x) - \phi_{l_0 + a + kd_2}(y) \|_\infty \\
\leq \sum_{k \leq 0} \frac{(1 + \varepsilon)^{2 + l_0 + a + kd_2}}{(1 + \varepsilon)^{(l_0 + a + kd_2)/2}} + \sum_{k > 0} \frac{(1 + \varepsilon)^{l_0 + 2}}{(1 + \varepsilon)^{(l_0 + a + kd_2)/2}} \\
= (1 + O(\varepsilon)) \sqrt{d_M(x, y)}
\]

Thus it follows (by a tedious calculation) that \(\|\phi(x) - \phi(y)\| \leq (1 + O(\varepsilon)) \sqrt{d_M(x, y)}\), see [HM05] for more details.

References
