A Generalization of the Assouad Embedding

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Abstract

Assouad [Ass83] showed that any metric $\mathcal{M}$ of doubling dimension dim can be embeddeded into l_∞^d where $d \leq \varepsilon^{-O(\dim)}$ with distortion at most $(1+\varepsilon)$. If $X \subseteq \mathcal{M}$ is of doubling dimension dim, we show that we can use the Assouad technique to embed X into l_∞^d such that the distances from $X \cup S$ to X are preserved where $S \subseteq \mathcal{M}$ is an arbitrary but fixed finite set.

1 Modification of Assoud's embedding

Assouad [Ass83] showed that any metric $\mathcal{M}$ of doubling dimension dim can be embeddeded into l_∞^d where $d \leq \varepsilon^{-O(\dim)}$ with distortion at most $(1+\varepsilon)$. Following the description in [HM05], we obtain the following result:

Theorem 1.1 *Given a metric space $(\mathcal{M}, d_\mathcal{M})$, a subset $X \subseteq \mathcal{M}$ such that X has doubling dimension* dim*, and a finite subset $S \subseteq \mathcal{M}$, we can map the metric $(X \cup S, d_\mathcal{M})$ into l_∞^d with $d \leq \varepsilon^{O(-\dim)}$ by a mapping Φ such that for any $x \in X \cup S$ and any $y \in X$ $1 - O(\varepsilon) \leq \frac{\|\Phi(x)-\Phi(y)\|}{\sqrt{d_\mathcal{M}(x,y)}} \leq 1 + O(\varepsilon)$.*

The following elementary lemma, which we mention without proof, is used in the proof below.

Lemma 1.2 *Let $(\mathcal{M}, d_\mathcal{M})$ be any metric space and let $S \subseteq \mathcal{M}$ be a set of points such that for each $x \in \mathcal{M}$, $d(x,S) = \inf\{d(x,y) | y \in S\}$ exists and is attained for some $y \in S$. Then $d(x,S)$ is 1-lipschitz.*

Proof: Given $r > 0$, we first show how to embed $(X \cup S, d_\mathcal{M})$ into $l_\infty^{d_1}$ with $d_1 \leq \varepsilon^{-O(\dim)}$ such that the following hold

- For any $x \in X \cup S$ and $y \in X$, $\left\|\phi^{(r)}(x) - \phi^{(r)}(y)\right\| \leq \min\{r, d_\mathcal{M}(x,y)\}$.
- For $x \in X \cup S, y \in X$ if $d_\mathcal{M}(x,y) \in [(1+\varepsilon)r, 2r)$ then $\left\|\phi^{(r)}(x) - \phi^{(r)}(y)\right\| \geq (1-\varepsilon)r$.

Let $N^{(r)}$ be an εr net of X. Suppose we color the points of $N^{(r)}$ such that any two points p, q with $d_\mathcal{M}(p,q) \leq 4r$ get colored differently. Since the metric $(\mathcal{M}, d_\mathcal{M})$ is of doubling dimension dim, $d_1 = (\frac{4}{\varepsilon})^{\dim} = \varepsilon^{-O(\dim)}$ colors are sufficient for this purpose. For each color i denote the set of points in $N^{(r)}$ with color i by C_i and define the value $\phi_i^{(r)}(x) = \max\{0, r - d_\mathcal{M}(x, C_i)\}$. It is easy to see that $\phi_i^{(r)}(x) \leq r$ and hence

$\left\|\phi^{(r)}(x) - \phi^{(r)}(y)\right\| \leq r$. If $d_{\mathcal{M}}(x,y) > r$ then clearly $\left|\phi_i^{(r)}(x) - \phi^{(r)}(y)\right| < d_{\mathcal{M}}(x,y)$ for each i. Let $d_{\mathcal{M}}(x,y) \leq r$ and i be a color. If $\phi_i^{(r)}(x) = 0 = \phi_i^{(r)}(y)$ or $\phi_i^{(r)}(x) = r - d_{\mathcal{M}}(x, C_i), \phi_i^{(r)}(y) = r - d_{\mathcal{M}}(y, C_i)$, then also $\left|\phi_i^{(r)}(x) - \phi^{(r)}(y)\right| \leq d_{\mathcal{M}}(x,y)$. The other two cases are similar and so we consider one of them where $\phi_i^{(r)}(x) = r - d_{\mathcal{M}}(x, C_i)$ and $\phi_i^{(r)}(y) = 0$. Let $d_{\mathcal{M}}(x, C_i) = d_{\mathcal{M}}(x,p)$ where $p \in C_i$. Notice that $r \leq d_{\mathcal{M}}(y, C_i) \leq d_{\mathcal{M}}(y,p)$ Then,

$$\begin{aligned}
\left|\phi_i^{(r)}(x) - \phi_i^{(r)}(y)\right| &= r - d_{\mathcal{M}}(x,p) \\
&\leq d_{\mathcal{M}}(y,p) - d_{\mathcal{M}}(x,p) \\
&\leq d_{\mathcal{M}}(x,y)
\end{aligned}$$

In all the cases $\left|\phi_i^{(r)}(x) - \phi^{(r)}(y)\right| \leq d_{\mathcal{M}}(x,y)$. Thus $\left\|\phi^{(r)}(x) - \phi^{(r)}(y)\right\| \leq d_{\mathcal{M}}(x,y)$.

The other property can be seen as follows. Suppose $d_{\mathcal{M}}(x,y) \in [(1+\varepsilon)r, 2r)$ and $y \in X$. There is a point p of the net $N^{(r)}$ such that $d_{\mathcal{M}}(y,p) \leq \varepsilon r$. Let i be the color of p. Thus $\phi_i^{(r)}(y) \geq (1-\varepsilon)r$. Now $d_{\mathcal{M}}(x,p) \geq r$ and for any other point q of color i, $d_{\mathcal{M}}(q,x) \geq 4r - (2+\varepsilon)r = (2-\varepsilon)r$ and so $d_{\mathcal{M}}(x, C_i) \geq r$. This means $\phi_i^{(r)}(x) = 0$ and so in this coordinate i, $\left|\phi_i^{(r)}(x) - \phi_i^{(r)}(y)\right| \geq (1-\varepsilon)r$ and therefore $\left\|\phi^{(r)}(x) - \phi^{(r)}(y)\right\| \geq (1-\varepsilon)r$.

The rest of the Assouad construction should follow verbatim, but we reproduce it here for completeness. We use the maps $\phi^{(r)}$ constructed above for various values of r to embed points in $X \cup S$ into $\mathbb{R}^{d_1 d_2}$ with the l_∞ metric for some number d_2 depending on ε. The exact dependence of d_2 of ε will be specified later. For each integer k, let $\phi_k(x)$ denote the mapping which maps x to the matrix with d_2 rows and d_1 columns where the $k \mod d_2$ row is the vector $\phi^{(1+\varepsilon)^k}(x)$ as above and the rest of the entries are zero. We define $\phi(x)$ as

$$\phi(x) = \sum_{k \in \mathbb{Z}} \frac{\phi_k(x)}{(1+\varepsilon)^{k/2}}$$

We now estimate the value of $\|\phi(x) - \phi(y)\|$. Let $l_0 \in \mathbb{Z}$ be the unique integer for which $d_{\mathcal{M}}(x,y) \in [(1+\varepsilon)^{l_0+1}, (1+\varepsilon)^{l_0+2})$. We notice that for the integer l_0 the map $\phi_{l_0}(z)$, which is just $\phi^{(1+\varepsilon)^{l_0}}(z)$ with more coordinates that are zero, has the scale $r = (1+\varepsilon)^{l_0}$ such that $d_{\mathcal{M}}(x,y) \in [(1+\varepsilon)r, 2r)$. We choose $d_2 = 8\varepsilon^{-1}\log(\varepsilon^{-1})$ in what follows. In the row of the matrix congruent to $l_0 \mod d_2$ the difference between $\phi(x)$ and $\phi(y)$ can be lower bounded as

$$\begin{aligned}
\left\|\sum_{k \in \mathbb{Z}} \phi_{l_0+kd_2}(x) - \phi_{l_0+kd_2}(y)\right\|_\infty &\geq \|\phi_{l_0}(x) - \phi_{l_0}(y)\|_\infty - \sum_{k<0} \|\phi_{l_0+kd_2}(x) - \phi_{l_0+kd_2}(y)\|_\infty \\
&\quad - \sum_{k>0} \|\phi_{l_0+kd_2}(x) - \phi_{l_0+kd_2}(y)\|_\infty \\
&\geq (1-\varepsilon)(1+\varepsilon)^{l_0/2} - \sum_{k>0} \frac{(1+\varepsilon)^{l_0+2}}{(1+\varepsilon)^{(l_0+kd_2)/2}} - \sum_{k<0} \frac{(1+\varepsilon)^{l_0+2+d_2k}}{(1+\varepsilon)^{(l_0+kd_2)/2}} \\
&= (1 - O(\varepsilon))\sqrt{d_{\mathcal{M}}(x,y)}
\end{aligned}$$

On the other hand, for any integer $a \in \{0, 1, \ldots, d_2 - 1\}$ we have for the rows with index congruent to $(l_0 + a) \mod d_2$

$$\begin{aligned}
\left\| \sum_{k \in \mathbb{Z}} \phi_{l_0+a+kd_2}(x) - \phi_{l_0+a+kd_2}(y) \right\|_\infty &\leq \sum_{k \leq 0} \|\phi_{l_0+a+kd_2}(x) - \phi_{l_0+a+kd_2}(y)\|_\infty \\
&\quad + \sum_{k > 0} \|\phi_{l_0+a+kd_2}(x) - \phi_{l_0+a+kd_2}(y)\|_\infty \\
&\leq \sum_{k \leq 0} \frac{(1+\varepsilon)^{2+l_0+a+kd_2}}{(1+\varepsilon)^{(l_0+a+kd_2)/2}} + \sum_{k>0} \frac{(1+\varepsilon)^{l_0+2}}{(1+\varepsilon)^{(l_0+a+kd_2)/2}} \\
&= (1+O(\varepsilon))\sqrt{d_{\mathcal{M}}(x,y)}
\end{aligned}$$

Thus it follows (by a tedious calculation) that $\|\phi(x) - \phi(y)\| \leq (1+O(\varepsilon))\sqrt{d_{\mathcal{M}}(x,y)}$, see [HM05] for more details. ■

References

[Ass83] P. Assouad. Plongements lipschitziens dans $\mathbf{R}^n$. *Bull. Soc. Math. France*, 111(4):429–448, 1983.

[HM05] S. Har-Peled and M. Mendel. Fast construction of nets in low dimensional metrics, and their applications. In *Proc. 21st Annu. ACM Sympos. Comput. Geom.*, pages 150–158, 2005. http://www.uiuc.edu/~sariel/papers/04/lipschitz.